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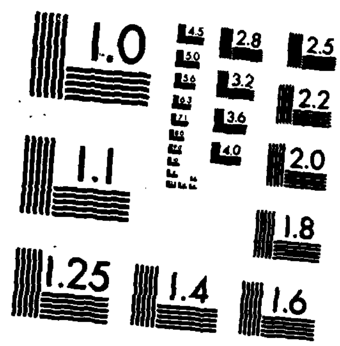

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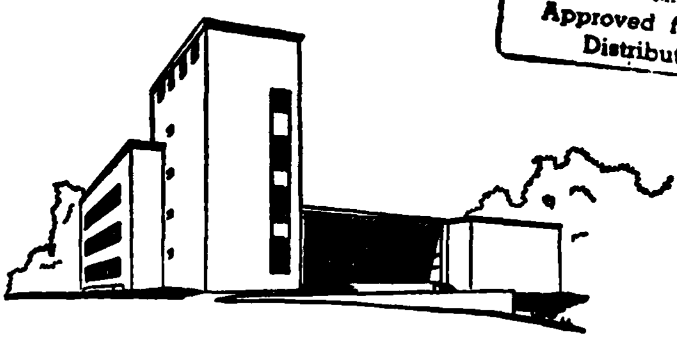

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GENERAL FACTORS IN GRAPHS

by

Gerard Cornuejols

July 1986

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General Factors in Graphs

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July 1986

Abstract: Consider a graph $G = (N,E)$ and, for each node $i \in N$, let B_i be a subset of $\{0,1,\ldots,d_G(i)\}$ where $d_G(i)$ denotes the degree of node i in G. The *general factor problem* asks whether there exists a subgraph of G, say $H = (N,F)$ where $F \subseteq E$, such that $d_H(i) \in B_i$ for every $i \in N$. This problem is NP-complete. A set B_i is said to have a *gap of length* $p \geq 1$ if there exists an integer $k \in B_i$ such that $k+1,\ldots,k+p \notin B_i$ and $k+p+1 \in B_i$. Lovàsz conjectured that the general factor problem can be solved in polynomial time when, in each B_i, all the gaps (if any) have length one. We prove this conjecture. In cubic graphs, the result is obtained via a reduction to the edge-and-triangle partitioning problem. In general graphs, the proof uses an augmenting path theorem and an Edmonds-type algorithm.

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1. Introduction

In this paper, we study a generalization of the classical factor problem. Given a graph G and a nonnegative integer b_i for each node i of G, the *factor problem* asks whether there exists a subgraph of G with exactly b_i edges incident with node i, for each i. This problem is well-solved. A polynomial algorithm is known (Edmonds and Johnson (1970)) as well as a powerful theorem to characterize the existence of solutions (Tutte (1952)).

The following generalization of the factor problem was studied by Lovàsz (1970b,1972). Let $G = (N,E)$ be a graph and, for each node $i \in N$, let B_i be a subset of $\{0, 1,..., d_G(i)\}$ where $d_G(i)$ denotes the degree of node i in G. The *general factor problem* asks whether there exists $F \subseteq E$ such that, for each node $i \in N$, the number of edges of F incident with i is an element of B_i. Some cases are known to be reducible to the classical factor problem, for example when B_i is an interval or a parity condition (Lovàsz (1970a,1972)). In section 2, we give conditions on B_i under which the general factor problem is reducible to the problem of partitioning the nodes of a graph into subsets that induce edges or triangles, a problem known to be polynomially solvable (Cornuéjols, Hartvigsen and Pulleyblank (1982)). As a consequence, we can solve the general factor problem in cubic graphs when $B_i \neq \{0,3\}$ for every $i \in N$.

The *antifactor problem* is the instance of the general factor problem where $|B_i| = d_G(i)$ for every $i \in N$, i.e. only one value is excluded at each node. The graphs that have an antifactor were characterized by Lovàsz (1973). These results have been generalized recently by Sebő (1986).

Lovàsz (1970b,1972) pointed out that the gaps in B_i play an important role in the study of the general factor problem. We say that B_i has a *gap of length* $p \geq 1$ if there exists an integer $k \in B_i$ such that $k+1,..., k+p \notin B_i$ and $k+p+1 \in B_i$. We allow a

given set B_i to have several gaps. Consider as examples the different instances of the general factor problem introduced above. If the set B_i is a simple factor condition or an interval then it has no gap, if B_i is a parity condition then all the gaps have length 1 and, if B_i is an antifactor condition, there is a unique gap of length 1. Lovàsz (1970b) characterized the solutions of the general factor problem when the sets B_i have no gap of length 2 or more. In section 3, we present a polynomial algorithm when this condition holds.

When gaps of length 2 or more are allowed, the general factor problem is NP-complete. Lovàsz and Plummer (1986) prove it using the NP-completeness of the 3-colorability of planar graphs. We close the introduction by giving another proof which also shows that the general factor problem is NP-complete even when G is cubic and bipartite. The *exact 3-cover problem* consists of a finite set K and a family $\mathcal{S}=\{S_i\}_{i=1,\ldots,m}$ of subsets of K such that $|S_i| = 3$ for $i = 1,\ldots, m$. The question is whether there exists $J \subseteq \{1,\ldots, m\}$ such that $\{S_j\}_{j \in J}$ induces a partition of K. To see this question as a general factor problem, define a bipartite graph G with a node n_k, for each element $k \in K$, a node n_S for each set $S \in \mathcal{S}$ and , for each $S = \{p,q,r\} \in \mathcal{S}$, edges joining n_S to the nodes n_p, n_q and n_r. In addition, define $B_i = \{0,3\}$ for the nodes associated with $S \in \mathcal{S}$, and $B_i = \{1\}$ for the nodes associated with $k \in K$. Now, G has a general factor if and only if $\mathcal{S}$ contains an exact 3-cover. In fact, it is known that the exact 3-cover problem is NP-complete even when each element of K belongs to exactly 3 sets of $\mathcal{S}$ (Garey and Johnson (1979)). This shows that the general factor problem is NP-complete even when the underlying graph G is cubic and bipartite.

2. Reduction to the edge-and-triangle partitioning problem

The *edge-and-triangle partitioning problem* asks the following question. Given a graph $G = (N,E)$ and a family T of *triangles* of G (complete graphs on 3 nodes), can the node set N be partitioned into sets of cardinality 2 or 3 so that each set of cardinality 2 induces an edge of E and each set of cardinality 3 induces a triangle of T. Of course, when $T = \emptyset$, this is the classical *1-factor problem*, i.e. the factor problem where $b_i = 1$ for all $i \in N$. Cornuéjols, Hartvigsen and Pulleyblank (1982) gave a polynomial algorithm to solve the edge-and-triangle partitioning problem. (As $|T|$ is always polynomial in $|N|$, the algorithm is polynomial in $|N|$.)

A seemingly unrelated problem is the following instance of the general factor problem. The graph $G = (N^{=} \cup N^{\neq}, E)$ is bipartite, $B_i = \{1\}$ for $i \in N^{=}$ and $B_i = \{0,2,3,\dots,d_G(i)\}$ for $i \in N^{\neq}$. We call this instance of the general factor problem the *bipartite 1-factor-antifactor problem.* In this paper we denote by $\delta_G(i)$ the set of edges of G incident with node i. With this notation the bipartite 1-factor-antifactor problem asks whether there exists $F \subseteq E$ such that $|F \cap \delta_G(i)| = 1$ for every $i \in N^{=}$ and $|F \cap \delta_G(i)| \neq 1$ for every $i \in N^{\neq}$. We call such an edge set F a *1-factor-antifactor.*

Lovàsz found a very nice reduction of the edge-and-triangle partitioning problem to the bipartite 1-factor-antifactor problem and asked whether, in general, the bipartite 1-factor-antifactor problem could be solved in polynomial time. Of course this question is a special case of Lovàsz's conjecture on the general factor problem without gaps of length greater than one. This question was communicated to me by Pulleyblank (1985). In the next theorem we show that, conversely, the bipartite 1-factor-antifactor problem can be polynomially reduced to the edge-and-triangle partitioning problem. As a consequence we obtain a positive answer to Lovàsz's question.

<u>Theorem 1</u> *The edge-and-triangle partitioning problem polynomially reduces to the bipartite 1-factor-antifactor problem (Lovàsz). Conversely, the bipartite 1-factor-antifactor problem polynomially reduces to the edge-and-triangle partitioning problem.*

<u>Proof</u> : Consider a graph $G = (N,E)$ and a family T of triangles of G. Lovàsz proposed to construct a bipartite graph $H = (N^{=} \cup N^{\neq}, D)$ as follows. Take $N^{=} \equiv N$ and $N^{\neq}$ as having a node n_t for each triangle $t \in T$ and a node n_e for each edge $e \in E$ which does not belong to any triangle of T. For each node $n_t \in N^{\neq}$ so defined, join n_t to the 3 nodes of $N^{=}$ which belong to t, and for each $n_e \in N^{\neq}$ join n_e to the 2 nodes of $N^{=}$ which belongs to e. This defines the edge set D of the bipartite graph H.

Now H has a 1-factor-antifactor if and only if G has an edge-and-triangle partition. Specifically, a 1-factor-antifactor F in H yields the following family P of subsets of $N \equiv N^{=}$. Two nodes $u, v \in N$ belong to the same subset of P if, in H, the edges of F incident with u and v have a common endpoint in $N^{\neq}$. As $B_i = \{1\}$ for $i \in N^{\neq}$, P is a partition. As $1 \notin B_i$ for $i \in N^{\neq}$, no set P has cardinality 1. In fact, by construction of H, the sets of P induce edges or triangles of G. Conversely an edge-and-triangle partition P yields a 1-factor-antifactor F of H as follows. If $\{u,v,w\} \in P$, include in F the 3 edges (u,n_t), (v,n_t) and (w,n_t) where t is the triangle induced by u, v and w. If $\{u,v\} \in P$, then either $e = (u,v)$ belongs to no triangle of T ; in this case include in F the edges (u,n_e) and (v,n_e) . Or e belongs to at least one triangle of T ; choose one, say $t \in T$, and include in F the edges (u, n_t) and (v, n_t).

Conversely, we prove that the bipartite 1-factor-antifactor problem can be polynomially reduced to the edge-and-triangle partitioning problem.

Let $H = (N^{=} \cup N^{\neq}, D)$ be a bipartite graph. Construct the graph $G = (N,E)$ as follows. The node set of G is $N \equiv N^{=}$. The edge set of G is induced by the pairs of nodes

of $N^{=}$ which have a common neighbor in $N^{\neq}$. Define the family T as containing the triplets of nodes of $N^{=}$ which have a common neighbor in $N^{\neq}$. We claim that G has an edge-and-triangle partition relative to the family T if and only if H has a 1-factor-antifactor F. Consider F. As earlier, define the family P as comprising those subsets of $N = N^{=}$ that are joined by edges of F to a common neighbor in $N^{\neq}$. As F is a 1-factor-antifactor, P is a partition which does not contain sets of cardinality 1. If every set of P has cardinality 2 or 3, P is an edge-and-triangle partition. Now consider $S \in P$ of cardinality greater than 3. Any partition of S into sets of cardinality 2 or 3 can be used in P instead of S, as these sets induce edges of E or triangles of T by construction of G. So again we obtain an edge-and-triangle partition of G. Conversely, assume that we have an edge-and-triangle partition P of G. Consider $S \in P$. By definition of G, there exists in H a node $n \in N^{\neq}$ adjacent to each node of S. Define F to include the edges (i,n) for $i \in S$. The resulting edge set F induces a 1-factor-antifactor in H. ♦

The reduction of Theorem 1 provides a curious relationship between the 1-factor-antifactor problem and the edge-and-triangle partitioning problem. The next *theorem shows that other instances of the general factor problem can also be reduced* to edge-and-triangle partitioning. The proof involves a different type of reduction, defined locally at each node.

A *gadget* consists of a graph $H = (V \cup \{u_1,\dots,u_k\}, L \cup \{e_1,\dots,e_k\})$ such that $\delta_H(u_j) = \{e_j\}$ for $j = 1,\dots,k$, and of a family T of triangles of H. Some examples will be given in Figure 1. Let $G = (N,E)$ be a graph where the general factor problem must be solved. Given a node $i \in N$, the gadget (H,T) is said to *represent* the general factor condition B_i if, with the above notation,

(2.1) $k = d_G(i)$ and

(2.2) for $J \subseteq \{1,\dots, k\}$, the graph $(V \cup \{u_j\}_{j \in J}, L \cup \{e_j\}_{j \in J})$ has an edge-and-triangle partition relative to the family T if and only if $|J| \in B_i$.

Given a gadget representing B_i, one can perform the following construction. Let $\delta_G(i) = \{e_1,\dots, e_k\}$. Replace the node i of G by a new graph (V,L) so that, after construction, the graph induced by $L \cup \{e_1,\dots,e_k\}$ is the graph H of the gadget. Using this construction one can transform an instance of the general factor problem into an edge-and-triangle partitioning problem if, for each node $i \in N$, there exists a gadget that represents the condition B_i. In the next theorem, we construct gadgets that represent various conditions B_i. Some of these statements are already known, but we include them here for completeness.

Theorem 2 *Each of the following general factor conditions can be represented by a gadget:*

(2.3) B_i *is an interval, i.e.* $B_i = \{p,\dots, p+r\}$ *for* $r \geq 0$,

(2.4) B_i *is the intersection of an interval with a parity condition,*

i.e. $B_i = \{p, p+2,\dots,p+2r\}$ *for* $r \geq 1$,

(2.5) $B_i = \{p, p+2, p+3, \dots, p+r\}$ *for* $r \geq 3$,

(2.6) $B_i = \{p, p+1, \dots, p+r-2, p+r\}$ *for* $r \geq 3$.

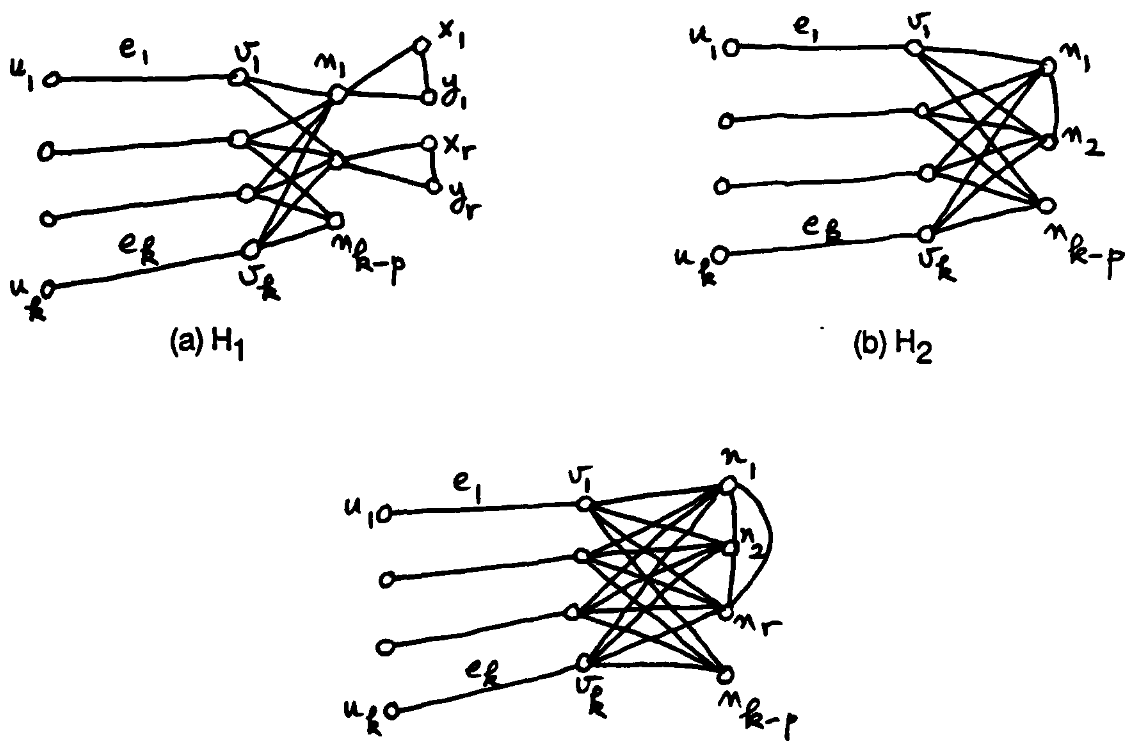
Proof: Let $\delta_G(i) = \{e_1,\dots,e_k\}$. The gadgets for (2.3)-(2.6) are based on the classical transformation of a b-factor node into 1-factor nodes. Let $e_j = (u_j,v_j)$ for $j = 1,\dots, k$, $V = \{v_j\}_{j=1,\dots,k} \cup \{n_t\}_{t=1,\dots,k-p}$ and $L = \{(v_j,n_t)\colon 1 \leq j \leq k \text{ and } 1 \leq t \leq k-p\}$, i.e. the graph (V,L) is a complete bipartite graph. Using the notation introduced earlier, the gadget $(H,\varnothing)$ represents the condition $B_i = \{p\}$.

To obtain (2.3), it suffices to attach r triangles (n_t,x_t,y_t) where x_t and y_t are new nodes, for $t = 1,\dots, r$ (see Figure 1a). If H_1 denotes this new graph and T_1 comprises these r triangles, then the gadget (H_1,T_1) represents (2.3).

To obtain (2.4), we add edges (n_{2t-1}, n_{2t}) for $t = 1, \dots, r$, to the bipartite graph H (see Figure 1b). If H_2 denotes this new graph, the gadget $(H_2, \varnothing)$ represents (2.4).

To obtain (2.5), we join the nodes $n_1, \dots, n_r$ by edges so that they form a clique of size r (see Figure 1c). If H_3 denotes this new graph and T_3 comprizes all the triangles of the clique $\{n_1, \dots, n_r\}$, the gadget (H_3, T_3) represents (2.5).

To obtain (2.6), it suffices to insert a node of degree 2 on each of the edges $e_1, \dots, e_k$ in the gadget (H_3, T_3), with the appropriate choice of p and r. ♦



(a) H_1 (b) H_2

(c) H_3

Figure 1 Gadgets representing conditions (2.3), (2.4) and (2.5).

A question posed by Pulleyblank is whether there exist conditions B_i that cannot be represented by a gadget. We do not consider this question here but simply mention that we could not find a gadget representing $B_i = \{0,1,3,4\}$ where i is a node of degree 4.

Next we consider the consequences of Theorem 2 for the general factor problem in cubic graphs. As noted in the introduction, this question is NP-complete if the set $B_i = \{0,3\}$ occurs for some of the nodes. In the next theorem we show that, in a sense, this is the only bad case for cubic graphs.

<u>Theorem 3</u> *The general factor problem in cubic graphs can be solved in polynomial time if* $B_i \neq \{0,3\}$ *for every node of the graph.*

<u>Proof :</u> In addition to the trivial factor condition $B_i = \emptyset$ and to the excluded condition $B_i = \{0,3\}$, there remain 14 possibilities:

(i) 10 interval conditions of type (2.3),

(ii) 2 parity conditions of type (2.4),

(iii) one condition of type (2.5), and

(iv) one condition of type (2.6).

So Theorem 3 follows as a corollary of Theorem 2. ♦

3. Solution of the general factor problem when there is no gap of length 2 or more

A general factor F in a graph G must satisfy

(3.1) $|F \cap \delta_G(i)| \in B_i$ for every node i of G.

An algorithmic approach for finding a general factor might be to relax the node requirements (3.1) and to construct a sequence of edge sets that approachs these conditions, in some defined sense. In this paper, we take a dual approach. The conditions (3.1) will be satisfied throughout the algorithm, but we relax "edge requirements", i.e. we allow that an edge $e = (u,v)$ belongs to F at one end, say u, but does not belong to F at the other end v. The algorithm resolves these infeasibilities one at a time or shows that no general factor exists. To make this approach precise, we construct a graph H from G as follows. On each edge of G we insert two new nodes, each of degree 2, so that each edge of G is split into three edges of H.

We denote by N_G the nodes of G, by N_F the new nodes, called *feasibility nodes* and by $N = N_G \cup N_F$ the nodes of H. The general factor problem in G is equivalent to the general factor problem in H where the conditions B_i are kept unchanged for $i \in N_G$ and $B_i = \{1\}$ for every node $i \in N_F$. Let E be the edge set of H. A *general matching* of H is an edge set $M \subseteq E$ such that

(3.2) $|M \cap \delta_H(i)| \in B_i$ *for every* $i \in N_G$,

and $|M \cap \delta_H(i)| = 0$ or 1 for every $i \in N_F$.

Since no two nodes of N_G are adjacent, it is easy to construct a general matching of H. A node $i \in N_F$ is *infeasible* if $|M \cap \delta_H(i)| = 0$. The *infeasibility* of a general matching M is the number of infeasible nodes of M. A general matching of H with an infeasibility equal to 0 corresponds to a general factor of G. In this section we present an algorithm which finds a general matching of H with smallest infeasibility.

Before stating the algorithm, we give a characterization of the general

matchings with smallest infeasibility. Let M be a general matching which does not have smallest infeasibility. We will show the existence of a certain type of path, called M-augmenting path, that yields an improvement of M by interchanging the edges in and out of M on the path. In this paper, paths may have repeated nodes but no repeated edges. Consider any path starting at a node $u_1 \in N_F$ such that $|M \cap \delta_H(u_1)| = 0$, say $P = (u_1, u_2, \ldots, u_p)$. For each node u_i of P, $i \geq 2$, define

(3.3) $\quad \beta_i = |M \cap \delta_H(u_i)| + |P_i| - |M_i|$

where M_i denotes the set of edges $(u_{j-1}, u_j) \in M$ that are incident with u_i, for $j \leq i$, and where P_i denotes the edges $(u_{j-1}, u_j) \notin M$ incident with u_i for $j \leq i$. In other words, β_i is the number of edges of M incident with u_i that would result if we interchanged the edges in and out of M on the subpath of P joining u_1 to u_i.

Let $l_i = \min\{b: b \in B_i\}$ and $m_i = \max\{b: b \in B_i\}$. In the remainder, we will write B_i, l_i and m_i instead of B_{ui}, l_{ui} and m_{ui} for notational simplicity.

An M-*augmenting path* $P = (u_1, u_2, \ldots, u_p)$ is defined by the following properties.

(3.4) $\quad u_1 \in N_F$ and $|M \cap \delta_H(u_1)| = 0$.

For every node u_i, $1 < i < p$,

(3.5) $\quad l_i \leq \beta_i + \varepsilon_i \leq m_i$ where $\varepsilon_i = -1$ if $(u_i, u_{i+1}) \in M$, 1 otherwise,

(3.6) $\quad \beta_i \notin B_i$.

(3.7) $\quad \beta_p \in B_p$.

<u>Theorem 4</u> *Let* M *be a general matching of* H. *The matching* M *has smallest infeasibility if and only if there exists no* M-*augmenting path.*

Proof: Given an M-augmenting path P, the general matching M can be improved into a general matching having a smaller infeasibility by interchanging the edges in and out

of M on the path P. For every node u_i of P that belongs to N_G, condition (3.2) still holds after the interchange as a consequence of (3.5), (3.6) and the fact that the gaps of B_i have length 1. In addition, every node of P that belongs to N_F is feasible after the interchange.

Conversely, assume that M does not have smallest infeasibility and, among the general matchings with a smaller infeasibility than M, choose one, say M', with the minimum number of nodes $u \in N_F$ such that $|M \cap \delta_H(u)| = 1$ and $|M' \cap \delta_H(u)| = 0$. As M' has a smaller infeasibility than M, there exists $u_1 \in N_F$ such that $|M \cap \delta_H(u_1)| = 0$ and $|M' \cap \delta_H(u_1)| = 1$. Let $D = M \Delta M'$, where Δ denotes the symmetric difference of two sets. We use the edges of D to construct a path $P = (u_1, u_2, ..., u_k)$ that satisfies conditions (3.5) and (3.6) for every $1 < i \leq k$. Now consider the different cases that may prevent us from pursuing the construction of P.

The first case is by violation of the lower bound in (3.5). This will occur when $(u_{k-1}, u_k) \in M$, $\beta_k = l_k - 1$ and, in D, every edge of M' already belongs to P. But this implies that $|M' \cap \delta_H(u_k)| < l_k$. Note that $u_k \in N_G$ would be a contradiction to the fact that M' is a general matching of H. So $u_k \in N_F$ and $|M' \cap \delta_H(u_k)| = 0$. By interchanging the edges in and out of M on the path P, we get a general matching M" with the same infeasibility as M', but with fewer nodes u such that $|M \cap \delta_H(u)| = 1$ and $|M'' \cap \delta_H(u)| = 0$, a contradiction to the choice of M'. Similarly, a violation of the upper bound in (3.5) would occur when $(u_{k-1}, u_k) \notin M$, $\beta_k = m_k + 1$ and, in D, every edge of M already belongs to P. This implies that $|M' \cap \delta_H(u_k)| > m_k$, a contradiction.

So eventually the construction of P must be stopped because a node u_p satisfying (3.7) is found. ♦

Given a general matching M, the search for M-augmenting paths is done by

growing an alternating forest. Some nodes of the forest may be shrunk critical subgraphs of H. A *critical subgraph* C is defined by the property that no general matching M of H satisfies

(3.8) $\quad |M \cap \delta_C(i)| \in B_i$ for every node i of C,

but, for any node i^* of C, there exists a general matching M such that (3.8) holds for every node $i \neq i^*$ and, in addition, $|M \cap \delta_C(i^*)| + 1 \in B_{i^*}$. It follows from the definition of C that, if M is a general factor of H, then there is at least one edge of M with one end in C and the other outside C.

In the course of the algorithm, it must be decided whether certain subgraphs of H are critical. When such a subgraph C is encountered, it already has the property that, for every node i^* of C, there exists a general matching such that (3.8) holds for every $i \neq i^*$ and $|M \cap \delta_C(i^*)| + 1 \in B_{i^*}$. In addition, an edge set M is available with this property for some given i^*. So the criticality of C actually reduces to the question of whether there exists an M-augmenting path in C, say $P = (u_1 = i^*, u_2, \ldots, u_p)$. Given u_p, this question can itself be reduced to two simple factor problems as explained below. Then, considering each node $u_p \in N_G$ in C as the potential final node of P, we can answer whether C is critical in polynomial time. Now assume that u_1 and u_p are given, and consider any node u_i of P such that $1 < i < p$. Conditions (3.5) and (3.6) imply the following relationships. Let $b = |M \cap \delta_H(u_i)|$. Consider the largest $j \leq b$ such that $j-1, j \in B_i$ or, if such a value j does not exist, such that $j-2, j-1 \notin B_i$. Let $s = (b-j)/2$. Note that s is an integer since all gaps have length one. The inequality $\beta_i + \varepsilon_i \geq b-2s$ must hold as a consequence of (3.5) and (3.6). Similarly, consider the smallest $k \geq b$ such that $k, k+1 \in B_i$ or, if such k does not exist, such that $k+1, k+2 \notin B_i$. Let $t = (k-b)/2$. Again t is integer and (3.5), (3.6) imply that $\beta_i + \varepsilon_i \leq b+2t$. In other words,

after the augmentation, the new general matching M' must have the property that $b-2s \le |M' \cap \delta_H(u_i)| \le b+2t$. Since P has an even number of edges incident with u_i, the condition at node u_i is the intersection of an interval with a parity condition. We have seen earlier how this can be polynomially transformed into 1-factor conditions (see Figure 1b). Note that no triangle is needed in this transformation. For the last node of the path P, say u_p, there are only two possibilities, namely $|M' \cap \delta_H(u_i)| = b-2s-1$ where $s = (b-j)/2$ and j is the largest integer such that $j \le b$ and $j-1, j \in B_i$. Or $|M' \cap \delta_H(u_i)| = b+2t+1$ where $t = (k-b)/2$ and k is the smallest integer such that $k \ge b$ and $k, k+1 \in B_i$. Each of these two factor conditions can be checked in turn and the question of the existence of an augmenting path P joining u_1 to u_p is therefore solvable in polynomial time.

We are now ready to state the algorithm for finding a general matching with minimum infeasibility in the graph H. Relative to any general matching M, the algorithm constructs a forest whose edges are alternately in and out of M (except possibly for pendant edges). The nodes of the alternating forest are either real nodes of H or shrunk critical subgraphs of H. Each tree of the forest has a *root* which is either an infeasible node of N_F or a shrunk node that contains an infeasible node of N_F. The nodes of the forest are called alternately *odd* and *even* on any path originating at the root, with the root node being even. Shrunk nodes of the alternating forest are always even. Every real even node i of the forest which is not a root satisfies $|M \cap \delta_H(i)| = l_i$. Every odd node i of the forest satisfies $|M \cap \delta_H(i)| = m_i$. Every edge of M incident with a node of the forest belongs to the forest.

Algorithm

Step 0 (Initialization) Start with any general matching M . Go to Step 1.

Step 1 (Optimality test) If M is a general factor, stop. Otherwise, start with the

infeasible nodes of N_F as the roots of the alternating forest. These nodes are even nodes of the forest. Go to Step 2.

Step 2 (Edge selection) Look for an edge which does not belong to the alternating forest and joins an even node of the forest to a node which is not an odd node of the forest. If no such edge exists, stop: there is no general factor (this claim will be proved later). Otherwise let e be an edge joining an even node u to a node v, where v is not an odd node of the forest.

Case 1 v is not in the forest, $v \in N_F$ and the node w defined by $(v,w) \in M$ is such that $w \in N_F$.(Note that the edge $(v,w) \in M$ exists as v is a feasible node of H.)Go to Step 3a.

Case 2 v is not in the forest, $v \in N_F$ and the node w defined by $(v,w) \in M$ is such that $w \in N_G$. If $|M \cap \delta_H(w)| = l_w$, go to Step 3b. If $|M \cap \delta_H(w)|-1 \in B_w$, go to Step 4a. Finally, if $|M \cap \delta_H(w)| > l_w$ and $|M \cap \delta_H(w)|-1 \notin B_w$, go to Step 3c.

Case 3 v is not in the forest and $v \in N_G$. If $|M \cap \delta_H(v)| = m_v$, go to Step 3d. If $|M \cap \delta_H(v)|+1 \in B_v$, go to Step 4a. Finally, if $|M \cap \delta_H(v)| < m_v$ and $|M \cap \delta_H(v)|+1 \notin B_v$, go to Step 3e.

Case 4 v is an even node of the forest and belongs to a different tree than u. Go to Step 4b.

Case 5 v is an even node of the forest and belong to the same tree as u. Consider the cycle closed by adding the edge e to the tree and define C to be the subgraph of H induced by the nodes of G in the cycle or within shrunk nodes of the cycle. Go to Step 5.

Step 3 (Growing the forest)

(a) Grow the alternating forest by adding the edges e and (v,w) to the forest, making v an odd node and w an even node. Go to Step 2.

(b) Let $x_1,\dots, x_k$ be the endpoints of the edges of M incident with w, other than the node v. Grow the alternating forest by adding the edges e, (v,w) and $(w,x_1),\dots, (w,x_k)$ to the

forest, making $v, x_1, \ldots, x_k$ odd nodes of the forest and w an even node. Go to Step 2.

(c) Let $x_1, \ldots, x_k$ be the endpoints of the edges of M incident with w, other than the node v. Grow the alternating forest by adding the edges e and (v,w), making v an odd node of the forest and shrinking the nodes $w, x_1, \ldots, x_k$ into an even node of the forest. Go to Step 2.

(d) Let $w_1, \ldots, w_k$ be the endpoints of the edges of M incident with v, where $k = m_v$. Grow the alternating forest by adding the edges e, $(v,w_1), \ldots, (v,w_k)$ to the forest, making v an odd node of the forest and making the nodes $w_1, \ldots, w_k$ even nodes of the forest. Go to Step 2.

(e) Let $w_1, \ldots, w_k$ be the endpoints of the edges of M incident with v. Shrink $u, v, w_1, \ldots, w_k$ into an even node the forest. Go to Step 2.

<u>Step 4</u> (Augmentation)

(a) Augment the general matching M by interchanging the edges in M and out of M on the path from v to the root of the tree containing v. Go to Step 1.

(b) Augment M by adding the edge e to M and by interchanging the edges in and out of M on the paths from u to the root of the tree containing u and from v to the root of the tree containing v. Go to Step 1.

Note that M can always be modified appropriatly within the shrunk nodes since they are critical.

<u>Step 5</u> (Augmentation or shrinking) Look for an augmenting path joining the root of the tree containing u and v to a node $u_p \in N_G$ in the set C. This can be performed in polynomial time as explained above. If such a path exists, augment M by interchanging the edges in and out of M on the path. Go to Step 1. If, for every $u_p \in N_G$ in C, no augmenting path exists, then shrink C into an even node of the alternating forest. Also shrink into the same even node every even node of the forest which is incident with a node of C and every odd node of degree 1 in the forest which is incident with a node of C. Go to Step 2. <u>End of Algorithm</u>

The algorithm terminates after at most $2 \mid N_F \mid$ augmentations. Between augmentations, Steps 3 and 5 are visited at most $\mid N \mid$ times. So the algorithm is polynomial. If the algorithm stops in Step 1, a general factor has been found. In order to prove the validity of the algorithm, we only have to show that, when it stops in Step 2, no general factor exists. Consider the set S comprising the odd nodes of the alternating forest at termination of the algorithm. A general factor can have at most $\Sigma\{m_v: v \in S\}$ edges joining nodes of S to nodes of N-S and the current M has just that many. In any general factor, at least one edge from S is required for each even node of the forest which is shrunk, and l_i such edges are required for each even node i which is a real node of H. The difference between the requirements from even nodes and the availability from the odd nodes is equal to the number of roots in the forest. Thus M leaves the smallest number of infeasible nodes in N_F. In particular, this shows that no general factor exists. Therefore, we get the following theorem.

Theorem 5 *The graph* H *has a general factor if an only if, for any* $S \subseteq N$,

$$\Sigma\{m_v: v \in S\} \geq \tau_S + \Sigma\{l_v: v \in L_S\},$$

where τ_S *is the number of connected components of* H(N-S) *which are critical, and* L_S *is the set of isolated nodes of* H(N-S).

This result is closely related to Theorem 4.3 of Lovàsz (1972). Extensions to a weighted version of the general factor problem are left for future research.

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Consider a graph $G = (N,E)$ and, for each node $i \in N$, let B_i be a subset of $\{0,1,\ldots,d_G(i)\}$ where $d_G(i)$ denotes the degree of node i in G. The general factor problem asks whether there exists a subgraph of G, say $H = (N,F)$ where $F \subseteq E$, such that $d_H(i) \in B_i$ for every $i \in N$. This problem is NP-complete. A set B_i is said to have a gap of length $p \geq 1$ if there exists an integer $k \in B_i$ such that $K+1,\ldots,k+p \notin B_i$ and $k+p+1 \in B_i$. Lovasz conjectured that the ———— (over)

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general factor problem can be solved in polynomial time when, in each B_i, all the gaps (if any) have length one. We prove this conjecture. In cubic graphs, the result is obtained via a reduction to the edge-and-triangle partitioning problem. In general graphs, the proof uses an augmenting path and an Edmonds-type algorithm.

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